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SHELL-STIFFENER INTERACTION. APPLICATION TO SIMPLY SUPPORTED CYLINDRICAL SHELLS UNDER UNIFORM PRESSURE

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Abstract—The limit analysis of a simply supported cylindrical shell with a central stiffener ring is dealt with in the case of an internal uniform pressure, with and without end load. Shell-stiffener interaction is studied, taking into account the stiffener effect in a discretized way, as a ring load.

1. INTRODUCTION

The limit analysis of circular cylindrical shells has been widely studied in the last few years, and many solutions of practical interest have been obtained[1, 2].

Such solutions, concerning homogeneous isotropic material, can be generalized for orthotropic material without significant difficulties. On the contrary, it is more difficult to take into account the "technological orthotropy" due to the use of stiffeners.

For limit analysis of stiffened shells, some results can be found in the literature[3–7], where the rib effect is usually distributed on the shell. This technique cannot furnish good results for important stiffeners.

In ref. [8] a different method is proposed, which takes into account the shell-stiffener interaction. In the present paper, an analogous method is applied to the case of a simply supported cylindrical shell, with a unique stiffening ring, where a rectangular cross-section is adopted. An internal uniform pressure is considered, in the absence of an end load.

To study the simultaneous plastic collapse conditions of both shell and rib, the effect of the stiffening ring is considered in a discretized way, as a ring load. Some results, which may be easily applied, are obtained by an analytical approach, having adopted Hodge's yield condition for the stress state in the shell and Tresca's yield condition for the plane stress state in the stiffener.

In the presence of an end load and a stiffener with a T cross-section, some extensions are also discussed.

2. PRELIMINARY ASSUMPTIONS

2.1 *Shell problems without an end load*

Circular cylindrical shells are considered here. Let L, R and t denote length, middle radius and thickness, respectively (Fig. 1). The usual assumptions of thin-shell theory regarding the small dimension of thickness and deflections are adopted. A homogeneous and orthotropic material, with rigid, perfectly plastic behavior, is considered. Let the only load be an internal uniform pressure P in the outward radial direction; no loads are considered in the axial direction. Material and loads, as well as geometry and supports, are axisymmetric. Then, if a cylindrical coordinate system (r, θ, X) with X along the symmetry axis is chosen, all functions will depend on X only.

Owing to the symmetry, the only displacement rate of the median surface of the shell is W. The stress state in the shell is defined by the axial bending moment M_x and

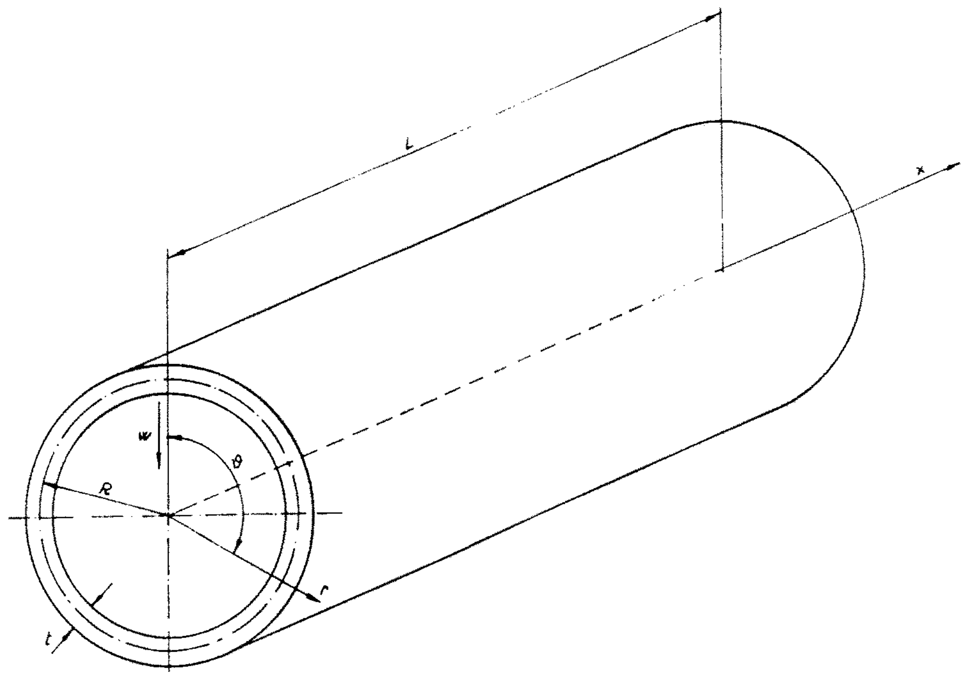

Fig. 1. Circular cylindrical shell.

the shear force S_x, as well as the circumferential normal force N_θ and the bending moment M_θ; positive directions are defined in Fig. 2.

The dissipation across a shell element, which does not depend on S_x and M_θ, is given by [1]:

$$D = -\frac{d^2W}{dX^2} M_x - \frac{W}{R} N_\theta. \tag{1}$$

Equilibrium conditions of the shell differential element provide

$$\frac{d^2M_x}{dX^2} + \frac{N_\theta}{R} - P = 0. \tag{2}$$

If $\sigma_{0\theta}$ and σ_{0x} denote yield stresses in the circumferential and axial directions, respectively, the following dimensionless variables can be defined:

$$x = \frac{X}{L}, \quad w = \frac{W}{R}$$
$$n = \frac{N_\theta}{N_0}, \quad m = \frac{M_x}{M_0}, \quad p = \frac{PR}{N_0} \tag{3}$$

where

$$N_0 = \sigma_{0\theta}\, t, \qquad M_0 = \frac{\sigma_{0x}\, t^2}{4}. \tag{4}$$

By denoting with a prime differentiation with respect to x, eqn (2) reads

$$m'' + 2\alpha^2(n - p) = 0, \tag{5}$$

where

$$\alpha^2 = \frac{2L^2}{Rt}\frac{\sigma_{0\theta}}{\sigma_{0x}}. \tag{6}$$

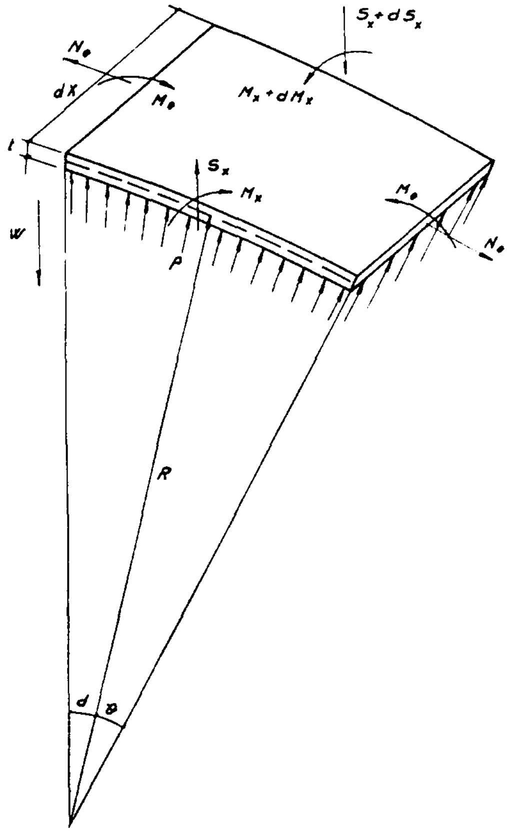

Fig. 2. Shell differential element.

Dimensionless parameter α combines structural geometry with the degree of material orthotropy.

Generalized strain rates corresponding to n and m are, respectively,

$$\kappa_1 = -w', \qquad \kappa_2 = -\frac{w''}{2\alpha^2}, \tag{7}$$

and (1) reads

$$D = \sigma_{0\theta}\, t(\kappa_2 m + \kappa_1 n). \tag{8}$$

The plastic yield condition can be expressed as depending on N_θ and M_x only, and by using dimensionless variables, it can be represented in an (n, m) plane[1].

The hexagonal condition of Fig. 3 (see also Table 1), which is the exact yield condition for a sandwich shell made of Tresca's material, is considered.

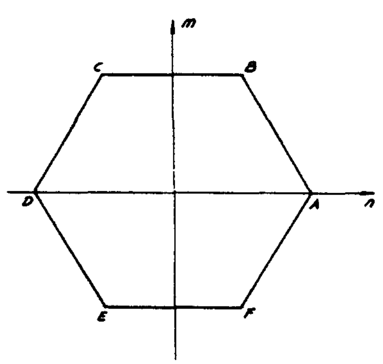

Fig. 3. Hodge's plastic yield condition.

Table 1. Hexagonal condition of Fig. 3

Side	Equation
AB	$2n + m - 2 = 0$
BC	$m - 1 = 0$
CD	$2n - m + 2 = 0$
DE	$2n + m + 2 = 0$
EF	$m + 1 = 0$
FA	$2n - m - 2 = 0$

If a shell element at the limit state is considered, the stresses at a point of the structure are represented by a point on the hexagon of Fig. 3. The shell stress state is represented by parts of the hexagon, i.e. by a stress profile. For each side of the yield condition, analytical expressions for m and n can be easily found, by integrating the equilibrium equation (see Table 2).

By virtue of the normality rule, analytical expressions of displacement rate w for each side AB, CD, DE and FA of the hexagon of Fig. 3 can be found on the basis of (3) (see Table 3). For sides BC and EF, the normality rule implies $w'' \neq 0$ and $w = 0$, which means that only particular points B, C, E and F are to be used as plastic regimes. These points correspond to plastic hinge circles in the shell, where w' is discontinuous.

2.2 *Shell-stiffener interaction*

Circular cylindrical shells, stiffened with circular ribs, are considered. It is assumed that a homogeneous and isotropic material, with rigid, perfectly plastic behavior, is used for stiffeners, and rectangular cross-section is adopted, the breadth and depth of which are denoted by b and d, respectively. For a general stiffener, a part of the shell of length λ is considered, limited by two sections in which shear forces S_x vanish (Fig. 4).

If σ_s denotes the radial stress between shell and rib (Fig. 5) and N_s is the circumferential normal force in the rib, the radial equilibrium condition of the shell provides

$$\sigma_s b \left(R + \frac{t}{2}\right) \mathrm{d}\theta + P\lambda R \,\mathrm{d}\theta - \int_\lambda N_\theta \,\mathrm{d}X \,\mathrm{d}\theta = 0 \tag{9}$$

or

$$\sigma_s = \frac{1}{bR_1} (N_\lambda - PR\lambda) \tag{10}$$

Table 2. Analytical expressions for m and n

Side	$n = n(x)$
AB	$C_1 \sinh \alpha x + C_2 \cosh \alpha x + p$
CD	$C_3 \sin \alpha x + C_4 \cos \alpha x + p$
DE	$C_5 \sinh \alpha x + C_6 \cosh \alpha x + p$
FA	$C_7 \sin \alpha x + C_8 \cos \alpha x + p$

Table 3. Analytical expressions of displacement rate w

Side	$w = w(x)$
AB	$D_1 \sinh \alpha x + D_2 \cosh \alpha x$
CD	$D_3 \sin \alpha x + D_4 \cos \alpha x$
DE	$D_5 \sinh \alpha x + D_6 \cosh \alpha x$
FA	$D_7 \sin \alpha x + D_8 \cos \alpha x$

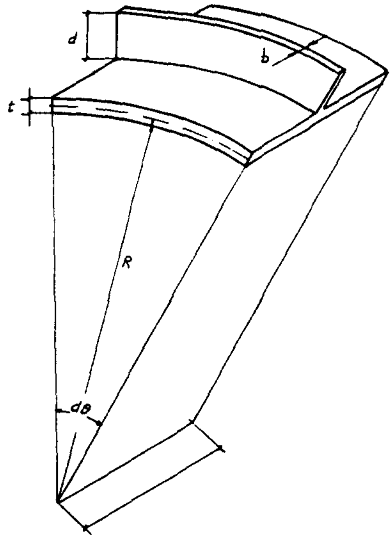

Fig. 4. Stiffener differential element.

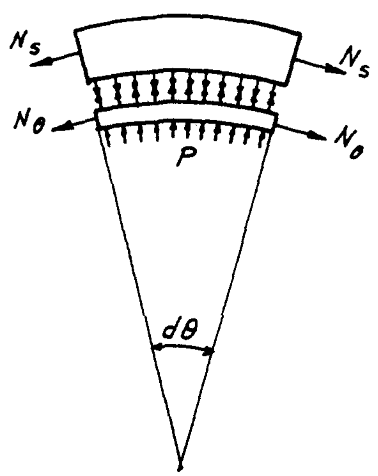

Fig. 5. Shell-stiffener interaction.

if

$$N_\lambda = \int_\lambda N_\theta \, dX, \qquad R_1 = R + \frac{t}{2} \tag{11}$$

are assumed.

For the stiffener, an axisymmetric state of plane stress in (r, θ) is considered. Equilibrium condition provides

$$\frac{d(\sigma_r r)}{dr} - \sigma_\theta = 0, \tag{12}$$

with boundary conditions

$$\begin{aligned} \sigma_r &= \sigma_s \quad \text{in} \quad r = R_1 \\ \sigma_r &= 0 \quad \text{in} \quad r = R_1 + d = R_2. \end{aligned} \tag{13}$$

Moreover, for resultant stresses N_λ and N_s, where

$$N_s = b \int_{R_1}^{R_2} \sigma_\theta \, \mathrm{d}r, \tag{14}$$

one has

$$\lambda RP = N_s + N_\lambda. \tag{15}$$

Elimination of P and N_λ from (10) and (15) provides

$$\sigma_s = -\frac{N_s}{bR_1}. \tag{16}$$

Taking into account (15) and boundary conditions of (13), $\sigma_\theta > 0$ and $\sigma_r < 0$ can be adopted for the stiffener in plane stress. Therefore, if Tresca's yield condition is considered, the plastic collapse requires

$$\sigma_\theta - \sigma_r = \sigma_{0s}, \tag{17}$$

and by integrating (12)

$$\sigma_\theta = \sigma_{0s}\left(1 + \ln \frac{r}{R_2}\right) \tag{18}$$

if σ_{0s} denotes the yield stress of the stiffener.

Then, from (14) and (15),

$$N_s = \sigma_{0s}\, b\, R_1 \ln \frac{R_2}{R_1} \tag{19}$$

$$P = \frac{1}{\lambda R}\left(\sigma_{0s}\, b\, R_1 \ln \frac{R_2}{R_1} + N_\lambda\right). \tag{20}$$

Using dimensionless variables that are defined in (3) and

$$n_\lambda = \frac{N_\lambda}{\sigma_{0\theta} t \, \lambda}, \tag{21}$$

(20) reads

$$p = n_\lambda + \frac{1}{\lambda} \frac{\sigma_{0s}}{\sigma_{0\theta}} \frac{bR_1}{t} \ln \frac{R_2}{R_1} \tag{22}$$

or

$$p = n_\lambda + \frac{L}{\lambda} s, \tag{23}$$

where

$$s = \frac{\sigma_{0s}}{\sigma_{0\theta}} \frac{bR_1}{Lt} \ln \frac{R_2}{R_1}. \tag{24}$$

The dimensionless parameter s represents geometric and physical characteristics of the stiffener that relate to those of the shell.

3. SIMPLY SUPPORTED SHELL WITHOUT STIFFENERS

In the hypotheses of Section 2, the case of a simply supported shell is considered (Fig. 6). Owing to the symmetry with respect to the central cross-section, only half of the shell is considered. Let L denote the length of this part. Some preliminary results can be found by studying the problem in the absence of stiffeners.

The simplest solution is obtained by adopting the plastic profile defined by the side AF of the hexagon of Fig. 3. Thus, (see Table 3)

$$n = C_7 \sin \alpha x + C_8 \cos \alpha x + p, \tag{25}$$

where x is measured from the support.

Using boundary conditions

$$\begin{aligned} n &= 1 \quad &\text{in} \quad x = 0 \\ n &= 0.5 \quad &\text{in} \quad x = 1 \\ n' &= 0 \quad &\text{in} \quad x = 1, \end{aligned} \tag{26}$$

constants of integration and limit pressure p can be found, yielding

$$n = (1 - p)\frac{\cos \alpha(1 - x)}{\cos \alpha} + p \tag{27}$$

$$p = 1 + \frac{0.5 \cos \alpha}{1 - \cos \alpha}. \tag{28}$$

The collapse mechanism associated with such a solution is (Table 3)

$$w = D_7 \sin \alpha x, \tag{29}$$

if relevant boundary conditions on w are taken into account. The mechanism involves a plastic hinge circle in $x = 1$, where the normality rule requires $w' > 0$. Thus, the solution is complete only for

$$\alpha < \frac{\pi}{2}. \tag{30}$$

For $\alpha = \pi/2$, (28) reads

$$p = 1. \tag{31}$$

For any $\alpha > \pi/2$, the collapse load is given by (31) and $n = 1$ [$\forall\, x \in (0, 1)$] can

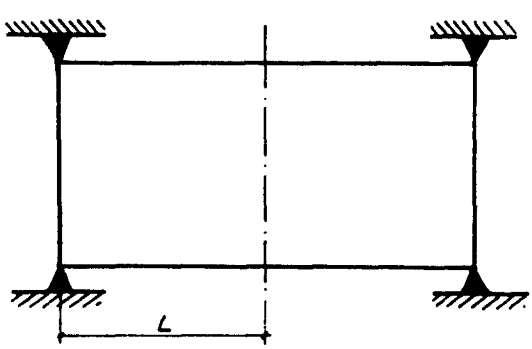

Fig. 6. Simply supported shell.

be found. The associated kinematic solution is

$$w = D_7 \sin \alpha x, \qquad x \in \left(0, \frac{\pi}{2\alpha}\right)$$
$$w = D_7, \qquad x \in \left(\frac{\pi}{2\alpha}, 1\right). \tag{32}$$

4. SIMPLY SUPPORTED SHELL WITH A CENTRAL STIFFENER

4.1 *Different solutions for* $\alpha \leq \pi/2$

(i) *Solution* 1. Let a central stiffener on the simply supported shell be considered (Fig. 7). The structural geometry is supposed to fulfill the condition $\alpha \leq \pi/2$.

Let the plastic profile be that shown in Fig. 8. Table 3 provides analytical expressions for displacement rate w:

$$w = D_7 \sin \alpha x + D_8 \cos \alpha x, \quad x \in (0, \bar{x})$$
$$w = D_7' \sin \alpha x + D_8' \cos \alpha x, \quad x \in (\bar{x}, 1). \tag{33}$$

From boundary conditions ($w = 0$ in $x = 0$ and $w' = 0$ in $x = 1$) and the continuity condition in $x = \bar{x}$, it follows that

$$w = D_7 \sin \alpha x, \qquad x \in (0, \bar{x})$$
$$w = D_7 \frac{\sin \alpha \bar{x}}{\cos \alpha(1 - \bar{x})} \cos \alpha(1 - x), \quad x \in (\bar{x}, 1). \tag{34}$$

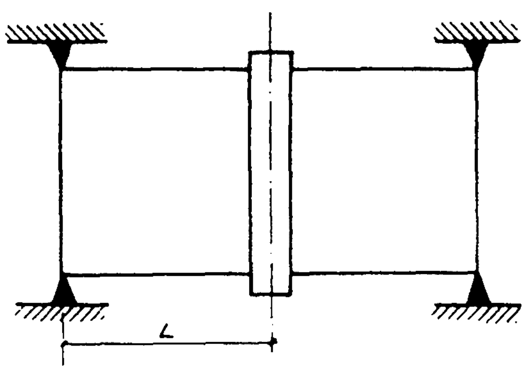

Fig. 7. Simply supported stiffened shell.

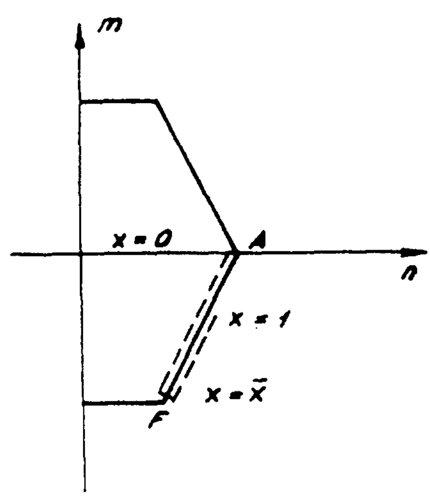

Fig. 8. Plastic profile for solution 1.

This solution is depicted in Fig. 9. The normality rule involves

$$\nu_1 \geq \nu_2, \tag{35}$$

i.e.

$$\text{ctg}\ \alpha(1 - \bar{x}) \geq \text{tg}\ \alpha\bar{x}, \tag{36}$$

which is fulfilled for any $\alpha < \pi/2$. Now, by using Table 2 and the relevant boundary conditions

$$\begin{aligned} n &= 1 \quad &&\text{in} \quad x = 0 \\ n = 0.5, \quad n' &= 0 \quad &&\text{in} \quad x = \bar{x}, \end{aligned} \tag{37}$$

the corresponding static solution can be found:

$$n = (1 - p)\frac{\cos \alpha(x - \bar{x})}{\cos \alpha\bar{x}} + p \tag{38}$$

$$p = 1 + \frac{0.5 \cos \alpha\bar{x}}{1 - \cos \alpha\bar{x}}. \tag{39}$$

The stiffener design can be defined by means of (11), (21) and (23) where $\lambda/L = 1 - \bar{x}$:

$$n_\lambda = \frac{1}{1 - \bar{x}}\int_{\bar{x}}^{1} n\, dx = \frac{1}{1 - \bar{x}}(1 - p)\frac{\sin \alpha(1 - \bar{x})}{\alpha \cos \alpha\bar{x}} \tag{40}$$

$$s = (1 - \bar{x})(p - n_\lambda) = -(1 - p)\frac{\sin \alpha(1 - \bar{x})}{\alpha \cos \alpha\bar{x}} = \frac{0.5 \sin \alpha(1 - \bar{x})}{\alpha(1 - \cos \alpha\bar{x})}. \tag{41}$$

Equations (39) and (41) provide the relationship between p and s, by means of parameter $\bar{x}$.

Note that the maximum value $\bar{x} = 1$ requires the stiffener parameter s to vanish. In this particular case, (39) reduces to (28).

An admissibility condition for solution 1 is given by the constraint

$$n \leq 1, \qquad \forall\, x \in (0, 1). \tag{42}$$

This implies, on the basis of (38) and (39),

$$\cos \alpha\bar{x} \leq \cos \alpha(x - \bar{x}), \qquad \forall\, x \in (0, 1), \tag{43}$$

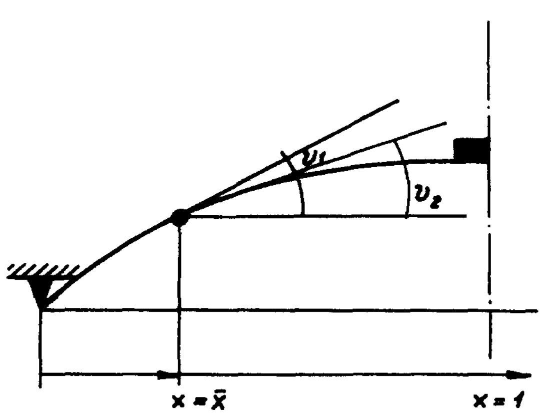

Fig. 9. Collapse mechanism for solution 1.

i.e.

$$\bar{x} \geq 0.5. \tag{44}$$

Corresponding limit values for p and s are

$$p = 1 + \frac{0.5 \cos \alpha/2}{1 - \cos \alpha/2} \tag{45}$$

$$s = \frac{0.5 \sin \alpha/2}{\alpha(1 - \cos \alpha/2)}. \tag{46}$$

(ii) *Solution* 2. If the rib stiffness exceeds the value given by (46), a different plastic regime is to be assumed (see Fig. 10). Function n for $x \in (0, x_1)$ and the load parameter p are given by (38) and (39). Parameter x_1 is defined by the condition

$$n = 1 \quad \text{in} \quad x = x_1, \tag{47}$$

which implies

$$\cos \alpha\bar{x} = \cos \alpha(x_1 - \bar{x}), \tag{48}$$

i.e.

$$x_1 = 2\bar{x}. \tag{49}$$

In $x \in (x_1, 1)$, one has (see Table 2)

$$n = C_1 \sinh \alpha x + C_2 \cosh \alpha x + p. \tag{50}$$

The integration constants C_1 and C_2 can be found by means of the continuity conditions in $x = x_1$:

$$\begin{aligned} C_1 \sinh \alpha x_1 + C_2 \cosh \alpha x_1 &= 1 - p \\ C_1 \cosh \alpha x_1 + C_2 \sinh \alpha x_1 &= (1 - p) \operatorname{tg} \alpha\bar{x}. \end{aligned} \tag{51}$$

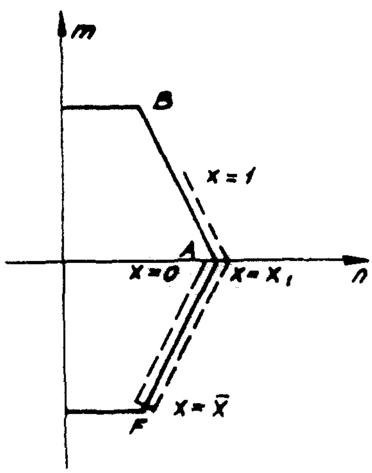

Fig. 10. Plastic profile for solution 2.

For designing the stiffener, one has

$$n_\lambda = \frac{1}{1-\bar{x}}\int_{\bar{x}}^{1} n\,dx = p + \frac{1}{\alpha(1-\bar{x})}(C_1\cosh\alpha + C_2\sinh\alpha)$$

$$= p + \frac{1-p}{\alpha(1-\bar{x})}[\sinh\alpha(1-x_1) + \mathrm{tg}\,\alpha\bar{x}\cosh\alpha(1-x_1)] \quad (52)$$

$$s = (1-\bar{x})(p-n_\lambda)$$

$$= \frac{1}{\alpha}\frac{0.5\cos\alpha\bar{x}}{1-\cos\alpha\bar{x}}[\sinh\alpha(1-x_1) + \mathrm{tg}\,\alpha\bar{x}\cosh\alpha(1-x_1)]. \quad (53)$$

By substituting the limit value $\bar{x} = 0.5$ in (53), (46) can be found, which verifies the continuity between solutions 1 and 2.

The adopted plastic profile also requires fulfilling the condition

$$1 \geqslant n \geqslant 0.5, \qquad \forall\, x \in (x_1, 1). \quad (54)$$

Hence, for the admissibility of solution 2, a minimum value is to be prescribed on $\bar{x}$. By means of the conditon $n(1) = 0.5$, this reads

$$\cos\alpha\bar{x}\cosh\alpha(1-2\bar{x}) + \sin\alpha\bar{x}\sinh\alpha(1-2\bar{x}) = 1, \quad (55)$$

and this minimum value $\bar{x} = \bar{x}_M$ can be found for any α.

To have a complete solution in the sense of limit analysis, a collapse mechanism is to be found.

Analytical forms of w are given in Table 3. The integration constants can be calculated, to within an arbitrary factor, by using boundary conditions [$w(0) = 0$, $w'(1) = 0$] and continuity conditions (for w in $x = \bar{x}$ and for w and w' in $x = x_1$). The collapse mechanism is depicted in Fig. 11.

The normality rule is verified for any $x \in (0, 1)$; the constraint $\nu_1 \geqslant \nu_2$ on plastic rotation in $x = \bar{x}$ implies

$$\mathrm{tgh}\,\alpha(1-x_1) + \mathrm{ctg}\,\alpha x_1 \geqslant 0, \quad (56)$$

which is fulfilled by any $\alpha < \pi/2$.

A particular case is found if $n_\theta(1) = 0.5$, because a second plastic hinge circle appears in $x = 1$. Boundary condition $w'(1) = 0$ is substituted by $w(1) = 0$, and the

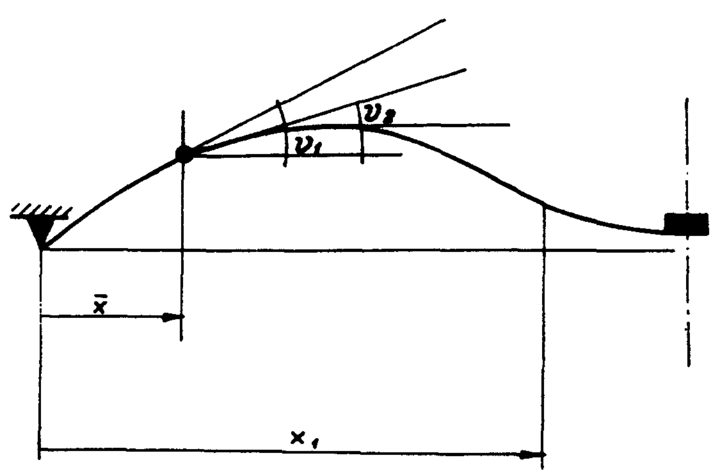

Fig. 11. Collapse mechanism for solution 2.

collapse solution does not require plasticity in the stiffener, for the stress field of which a statically admissible solution is to be found.

(iii) *Concluding remarks.* For a given $\alpha < \pi/2$, a plastic collapse solution can be obtained by previously selecting a value $\bar{x}$. For $\bar{x} \in (0.5, 1)$, parameters p and s are calculated by using (39) and (41); for $\bar{x} \leq 0.5$, in accordance with (55), $\bar{x} \geq \bar{x}_M$ is to be adopted and the solution is given by (39) and (53).

4.2 *Different solutions for* $\alpha \geq \pi/2$

(i) *Solution* 3. In the case $\alpha \geq \pi/2$, a different solution is to be studied. In particular, the plastic profile shown in Fig. 12 is adopted.

On the basis of Table 3 and relevant boundary conditions, one has

$$\begin{aligned} w &= D_7 \sin \alpha x, && x \in (0, x_1) \\ w &= D_7 \sin \alpha x_1 \frac{\cosh \alpha(1 - x)}{\cosh \alpha(1 - x_1)}, && x \in (x_1, 1), \end{aligned} \tag{57}$$

where the geometric parameter x_1 is given by

$$\operatorname{tgh} \alpha(1 - x_1) + \operatorname{ctg} \alpha x_1 = 0. \tag{58}$$

To have $w < 0$ for any $x > 0$, the solution of eqn (58) is to be chosen in such a way that

$$x_1 < \frac{\pi}{\alpha}. \tag{59}$$

From the static point of view, Table 2 provides

$$\begin{aligned} n &= C_7 \sin \alpha x + C_8 \cos \alpha x + p, && x \in (0, x_1) \\ n &= C_1 \sinh \alpha x + C_2 \cosh \alpha x + p, && x \in (x_1, 1). \end{aligned} \tag{60}$$

As x_1 is given by the kinematic solution, boundary conditions ($n = 1$ in $x = 0$ and $x = x_1$) provide integration constants C_7 and C_8:

$$C_7 = (1 - p) \operatorname{tg} \frac{\alpha x_1}{2}, \qquad C_8 = 1 - p. \tag{61}$$

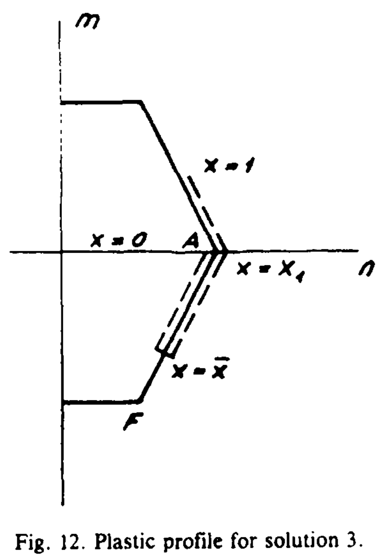

Fig. 12. Plastic profile for solution 3.

The values of C_1 and C_2 can be calculated from continuity conditions in $x = x_1$:

$$C_1 \sinh \alpha x_1 + C_2 \cosh \alpha x_1 = 1 - p$$
$$C_1 \cosh \alpha x_1 + C_2 \sinh \alpha x_1 = (1 - p) \operatorname{tg} \frac{\alpha x_1}{2}. \tag{62}$$

The abscissa $x = \bar{x}$, where shear forces S_x vanish, corresponds to the minimum value of n in $x \in (0, x_1)$. Condition $n'(\bar{x}) = 0$ provides

$$\bar{x} = \frac{x_1}{2}. \tag{63}$$

Therefore [see (38), (49) and (51)], for a given value of the load parameter p, the stiffener design corresponding to solution 3 can be calculated by using (52):

$$\begin{aligned} s &= (1 - \bar{x})(p - n_\lambda) \\ &= \frac{p - 1}{\alpha} [\sinh \alpha(1 - x_1) + \operatorname{tg} \alpha\bar{x} \cosh \alpha(1 - x_1)]. \end{aligned} \tag{64}$$

Note that for a shell without stiffener ($s = 0$), (64) gives $p = 1$, in accordance with the results of Section 3.

(ii) *Condition of existence for solution* 3. Stress fields of solution 3, defined in (60), involve admissibility conditions for the existence of the solution

$$n \geq 0.5 \quad \text{in} \quad x = \bar{x} \tag{65}$$

$$n \geq 0.5 \quad \text{in} \quad x = 1, \tag{66}$$

which read, respectively,

$$p \leq 1 + \frac{0.5 \cos \alpha\bar{x}}{1 - \cos \alpha\bar{x}} \tag{67}$$

$$p \leq 1 + \frac{0.5}{\cosh \alpha(1 - x_1) + \operatorname{tg} \alpha\bar{x} \sinh \alpha(1 - x_1) - 1}. \tag{68}$$

Accordingly, upper bounds on the stiffener parameter s can be calculated by using (64).

Hence, for a given $\alpha > \pi/2$, the corresponding x_1 is calculated by means of (58), and a value for the load parameter p can be selected in complying with (67) and (68). An examination of numerical solutions shows the first constraint is active for $\alpha \leq 2.424$, the second one for $\alpha \geq 2.424$. In both cases, if the maximum value is chosen for p and s, the collapse mechanism defined in solution 3 is to be modified.

In particular, for $\alpha = 2.424$ (two constraints active at the same time), two plastic hinge circles appear, in $x = \bar{x}$ and $x = 1$, and the collapse solution does not require plasticity in the stiffener.

If, for $\alpha < 2.424$, (67) is satisfied as an equality [i.e. p is given by (39)], a plastic hinge circle appears in $x = \bar{x}$ and solution 2 is to be used.

Since it is $\alpha > \pi/2$, the value of x_1 is to be chosen in complying with the upper bound defined by (56). The lower bound on x_1 is provided by (55).

Verification of (56) as an equality [i.e. eqn (58)] provides the continuity between solutions 3 and 2, for $\alpha > \pi/2$ (plastic hinge circle in $x = \bar{x}$ not active). Decreasing values of x_1 (with $x_1 \geq 2\bar{x}_M$) provide more important values of the load parameter p and the stiffener parameter s. The case $x_1 = 2\bar{x}_M$ involves the existence of two plastic hinge circles.

(iii) *Solution* 4. If (68) is satisfied as an equality ($\alpha \geq 2.424$), a different plastic regime is to be studied (Fig. 13).

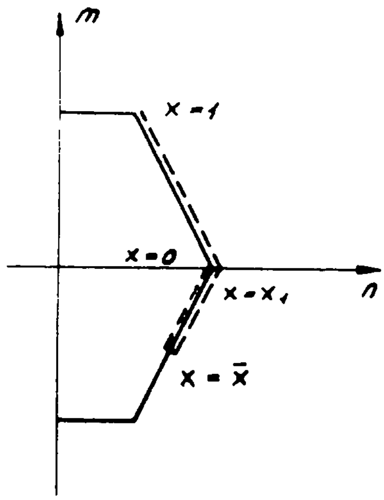

Fig. 13. Plastic profile for solution 4.

The static solution is defined by (60)–(64), and by (68) written as an equality, which provides the load parameter p for any x_1. The condition $n(\bar{x}) > 0.5$ implies (67).

From the kinematic point of view, taking into account the boundary condition in $x = 0$ and continuity conditions in $x = x_1$, one has

$$\begin{aligned} w &= D_7 \sin \alpha x, & x \in (0, x_1) \\ w &= D_7 [\cos \alpha x_1 \ \sinh \alpha(x - x_1) + \sin \alpha x_1 \cosh \alpha(x - x_1)], & x \in (x_1, 1). \end{aligned} \tag{69}$$

The condition $w < 0$ for any $x \in (0, 1)$ involves

$$x_1 < \frac{\pi}{\alpha} \tag{70}$$

$$\cos \alpha x_1 \ \sinh \alpha(1 - x_1) + \sin \alpha x_1 \cosh \alpha(1 - x_1) > 0 \tag{71}$$

because $D_7 < 0$ is to be adopted. Simultaneously verifying (67) and (71) is equivalent to the satisfaction of (70).

By means of numerical computations, the upper bound on x_1 is shown to be provided by (67) for $\alpha < 3.276$. If this static constraint is active, a solution with two plastic hinge circles is found, which does not require plasticity in the stiffener, and the maximum value of p, for any given α, is obtained. The continuity between solutions 3 and 4 is also verified.

For $\alpha > 3.276$, (71) is to be taken into account. Verifying this relation as an equality corresponds to condition $w(1) = 0$, which involves a new solution without plasticity in the stiffener.

In such cases, the solution is complete in the sense of limit analysis if a statically admissible solution is found for the stress field in the stiffener.

(iv) *Concluding remarks.* For a given $\alpha > \pi/2$, the solution can be constructed by using (57)–(64), if a load parameter p is assumed in such a way that both (67) and (68) are fulfilled (solution 3). Otherwise, solution 2 for $\alpha < 2.424$ and solution 4 for $\alpha > 2.424$ can be used.

The different relationships $p = p(s)$ are summarized in Fig. 14.

5. TAKING END LOADS INTO ACCOUNT

If one considers end loads due to internal uniform pressure, and N_x denotes the normal stress resultant in the axial direction X, the equilibrium condition along X provides

$$N_x = \frac{PR}{2}. \tag{72}$$

By using the dimensionless variable

$$n_x = \frac{N_x}{\sigma_{0x} t}, \tag{73}$$

eqn (72) reads

$$n_x = \frac{p}{2} \tag{74}$$

if $\sigma_{0x} = \sigma_{0\theta}$ is assumed.

As load parameter p fulfills the relation

$$p \geq 1, \tag{75}$$

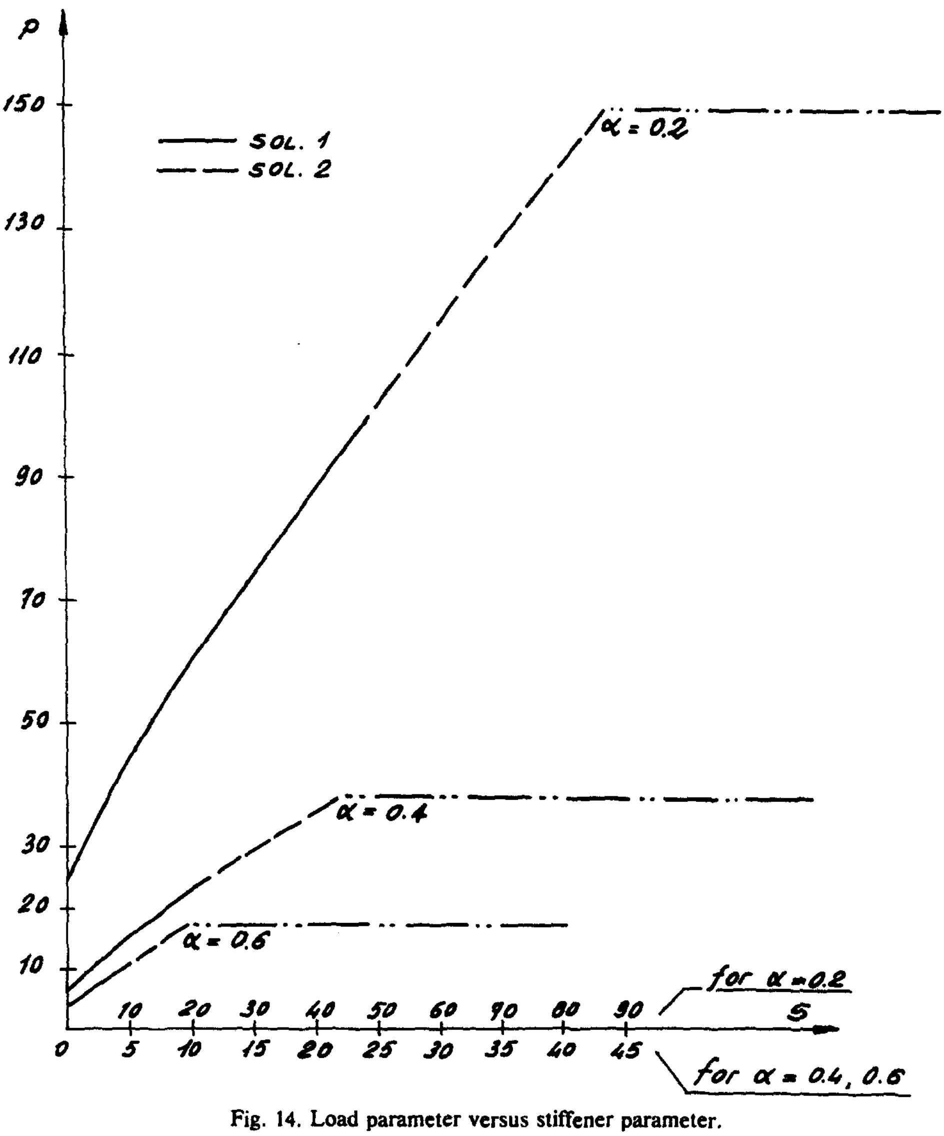

Fig. 14. Load parameter versus stiffener parameter.

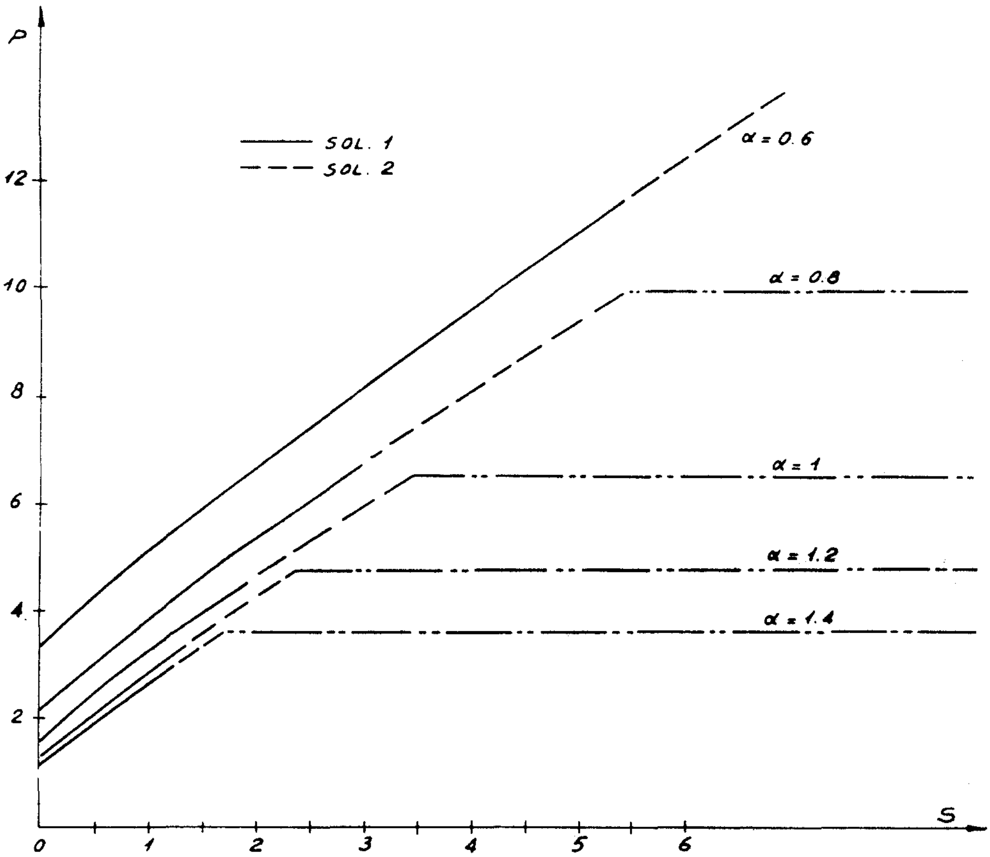

Fig. 14. (*Continued*)

one has

$$0.5 \leq n_x \leq 1. \quad (76)$$

The plastic yield condition can be represented[1] in the plane (n, m) by the rectangle shown in Fig. 15. Then

$$n = 1, \qquad \forall\, x \in (0, 1), \quad (77)$$

and for the stiffener design [see (21)–(24)]

$$n_\lambda = \frac{L}{\lambda} \int_{\lambda/L} n \, dx = 1 \quad (78)$$

$$s = \frac{\lambda}{L}(p - 1). \quad (79)$$

The moment function m can be found by integrating the equilibrium condition of (5), having taken into account (77):

$$m = -\alpha^2(1 - p) + C_1 x + C_2. \quad (80)$$

Let the plastic profile be adopted, as shown in Fig. 16. Boundary conditions on m are

$$\begin{aligned} m &= 0 \qquad \text{in} \quad x = 0 \\ m &= -(1 - n_x), \qquad m' = 0 \qquad \text{in} \quad x = \bar{x}, \end{aligned} \quad (81)$$

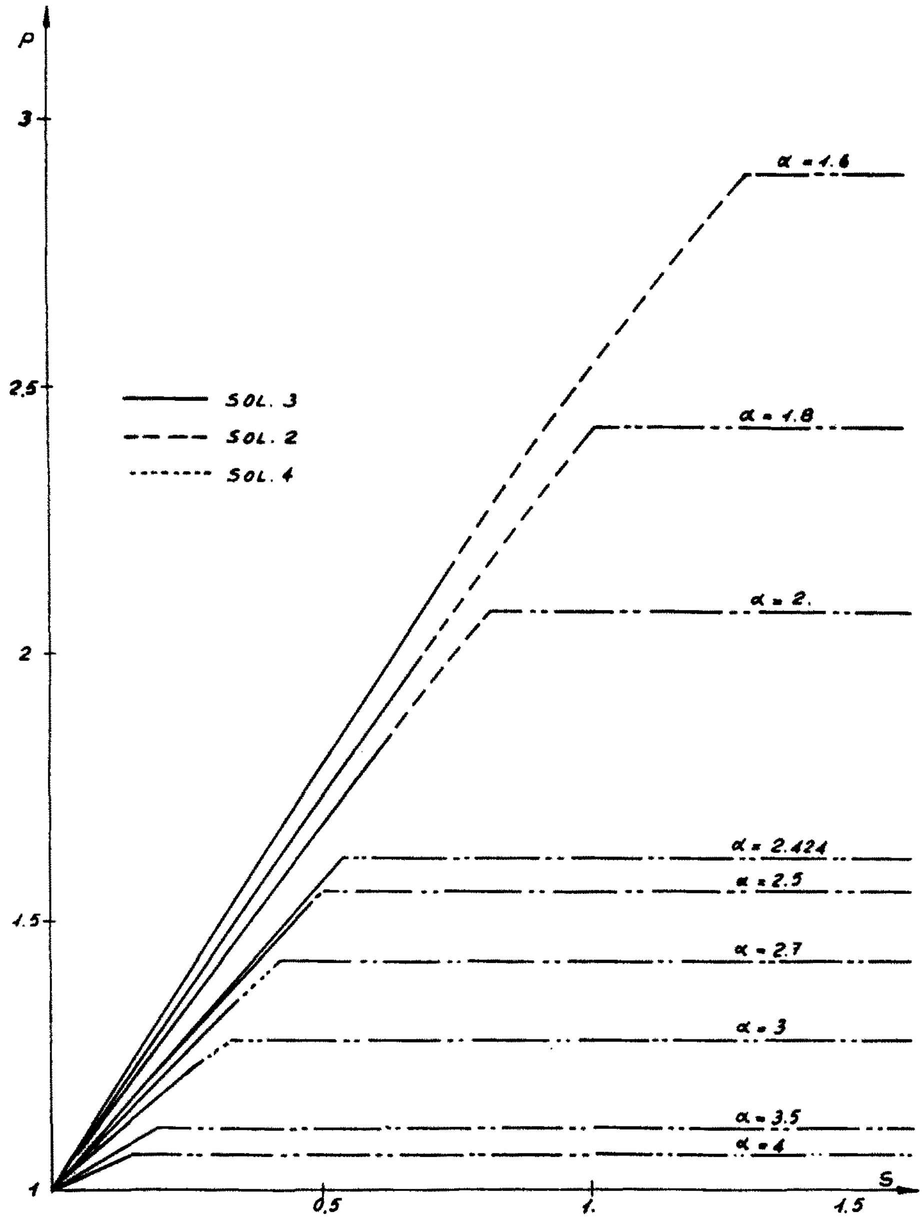

Fig. 14. (*Continued*)

and integration constants as well as the load parameter p can be calculated, giving

$$m = -(1 - n_x)\frac{x}{\bar{x}}\left(2 - \frac{x}{\bar{x}}\right) \tag{82}$$

$$p = 1 + \frac{1}{1 + 2\alpha^2\bar{x}^2}. \tag{83}$$

Accordingly, the stiffener parameter s is found:

$$s = \frac{1 - \bar{x}}{1 + 2\alpha^2\bar{x}^2}. \tag{84}$$

For $\bar{x} = 1$, the rib stiffness vanishes and (83) reads

$$p = 1 + \frac{1}{1 + 2\alpha^2}, \tag{85}$$

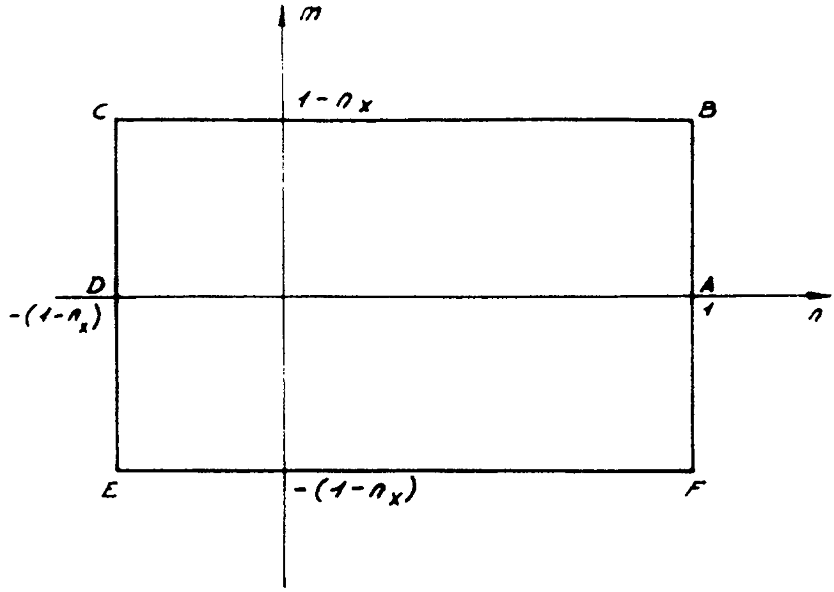

Fig. 15. Plastic yield condition for axially loaded shell.

which provides the plastic collapse load of an unstiffened simply supported shell, subjected to an internal uniform pressure, if end loads are taken into account.

From the constraint $m(1) \leq 1 - n_x$, the lower bound on $\bar{x}$ is found:

$$\bar{x} \geq 0.4142, \tag{86}$$

and the corresponding limit values for p and s are

$$p = 1 + \frac{1}{1 + 0.3431\ \alpha^2} \tag{87}$$

$$s = \frac{1}{1 + 0.3431\ \alpha^2}. \tag{88}$$

Equation (88) gives the minimum value of s to have a rigid stiffener; (87) also represents the plastic collapse load of a simply supported, clamped shell of length L.

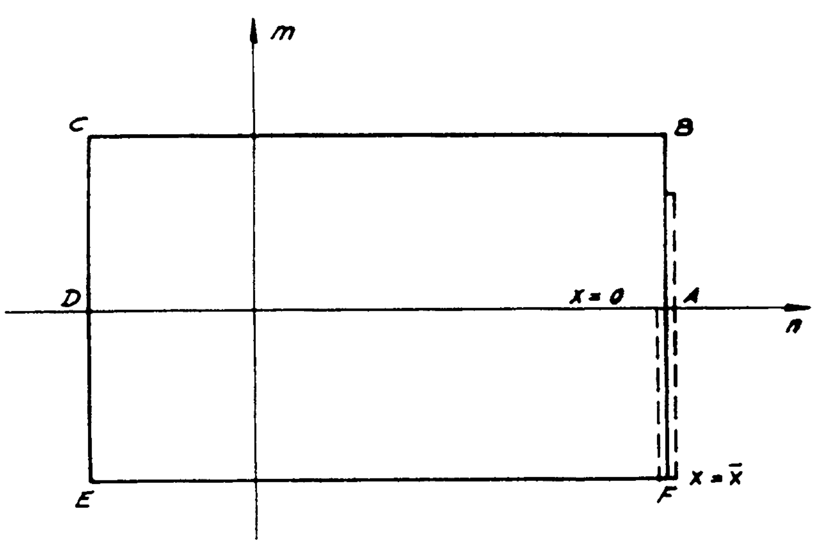

Fig. 16. Plastic profile for axially loaded shell.

From the kinematic point of view, the normality rule requires linear functions for displacement rate w. Therefore, in the general case $(0.4142 < \bar{x} \leq 1)$, one has

$$w = D_1 x, \quad x \in (0, \bar{x})$$
$$w = D_1 \bar{x}, \quad x \in (\bar{x}, 1). \tag{89}$$

The limit value $\bar{x} = 0.4142$ involves

$$w = D_1 x, \quad x \in (0, \bar{x})$$
$$w = D_1 \bar{x} \frac{1 - x}{1 - \bar{x}}, \quad x \in (\bar{x}, 1). \tag{90}$$

6. EXTENSION TO STIFFENERS WITH A T CROSS-SECTION

The results seen in the previous sections can be extended to the case of stiffeners with a T cross-section[8]. In particular, the equilibrium conditions of (9), (12) and (15) can be assumed for the web of the stiffener, whereas the boundary conditions of (13) are to be suitably modified because of the presence of the flange. If Tresca's yield condition is adopted, the web plane stress in plastic collapse conditions is characterized by (17), and the solutions seen in Section 4, for a stiffener with rectangular cross-section, can be used for the web collapse, by suitably modifying the form of the dimensionless parameter s, defined in (24).

The stress state of the flange can be characterized by the circumferential normal force and the bending moment, in the same way as the resultant stresses N_θ and M_x, for the shell element (see Section 2.1). The collapse conditions can be defined according to the same hexagonal yield condition of Fig. 3. In [8] and [9], two different collapse solutions are proposed for the flange, which are complete in the sense of limit analysis.

7. CONCLUSIONS

By taking into account rib effects in a discretized way as ring loads, the problem of limit analysis of simply supported shells with a central stiffener has been studied in detail. The rather complete presentation of the results can be used in applications, to obtain rational, even if not optimal, designs.

In particular, such solutions can be conceived that exhibit simultaneous collapse of shell and stiffener.

Furthermore, if analogous results for clamped shells[8] are used, rational designs of clamped or simply supported shells can be found in the presence of several stiffeners. The position of the stiffeners can be chosen in such a way that every part of the shell has the same collapse load, and the minimum dimension is adopted for the stiffeners, to have rigid stiffeners at the collapse of the shell.

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